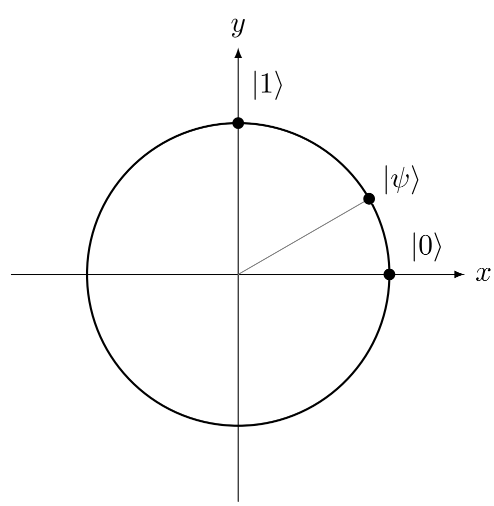
CMPT 476/981: Introduction to Quantum Algorithms
Assignment 3

Due **February 15th, 2024 at 11:59pm on coursys**
Complete individually and submit in PDF format.

Question 1 [4 points]: Projectors

Let $|\psi\rangle$ be a unit vector in $\mathbb{C}^d$ and $|\psi^\perp\rangle$ be a unit vector which is orthogonal to $|\psi\rangle$.

1. Let $P = |\psi\rangle\langle\psi|$. Compute $(I - 2P)|\psi\rangle$ and $(I - 2P)|\psi^\perp\rangle$.
2. Show that $(I - 2|\psi\rangle\langle\psi|)$ is unitary whenever $|\psi\rangle$ is a unit vector.
3. Suppose a single qubit has state $|\psi\rangle \in \mathbb{R}^2$ — that is, $|\psi\rangle$ is a unit vector in $\mathbb{R}^2$ where $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ can be viewed as the unit vector along the positive x-axis, and $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ the unit vector along the positive y axis. This is the two-dimensional picture of a quantum state which we've used in class:

   What is the geometric interpretation of the transformation $I - 2|0\rangle\langle 0|$ in $\mathbb{R}^2$?
4. Does the transformation $I - 2|0\rangle\langle 0|$ have a similar geometric interpretation in the Bloch sphere? Why or why not?

Question 2 [3 points]: Parity measurement

1. How is a parity measurement of two qubits different from measuring both bits in the computational basis **and then taking their parity**?

2. Devise a circuit using $CNOT$ gates and computational basis measurement which measures the parity of two qubits **without measuring either qubit itself**.

   Hint: you will need to use an *ancilla* — i.e. an additional qubit initialized to $|0\rangle$:

Question 3 [1 points]: Mixed states

Calculate the density matrix of the following ensembles.

1. $\{(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|+\rangle, 1)\}$
2. $\{(|0\rangle, \frac{1}{2}), (|+\rangle, \frac{1}{2})\}$
3. $\{(|00\rangle, \frac{1}{2}), (|01\rangle, \frac{1}{4}), (|10\rangle, \frac{1}{4})\}$

Question 4 [1 point]: Partial trace

Calculate the following reduced density matrix, taking A to be the first qubit (i.e. trace out the first qubit):

$$\mathrm{Tr}_A\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

Question 5 [3 points]: Positivity of the density operator

An operator A is *positive-semidefinite* if $\langle v|A|v\rangle$ is real and non-negative for any vector $|v\rangle$ of appropriate dimension. That is, A is positive-semidefinite if and only if $\langle v|A|v\rangle \in \mathbb{R}^+$ where $\mathbb{R}^+$ are the non-negative real numbers for all vectors $|v\rangle$.

Show that the density matrix $\rho = \sum_i p_i|\phi_i\rangle\langle\phi_i|$ of an ensemble of pure states $\{(|\phi_i\rangle, p_i)\}$ is a positive-semidefinite operator.

Question 6 [4 points]: No-communication

Suppose Alice and Bob share some mixed state ρ on a bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Recall that the partial measurement of Alice's qubit in basis $\{|e_i\rangle\}$ corresponds to the projective measurement $\{P_i = |e_i\rangle\langle e_i| \otimes I\}$ which maps $\rho \mapsto \sum_i P_i \rho P_i$.

Show that Bob's reduced density matrix is not affected by Alice measuring her qubit in any basis $\{|e_i\rangle\}$ of $\mathcal{H}_A$. Note: It may be helpful to assume that $\mathcal{H}_B$ has a basis $\{|f_j\rangle\}$.

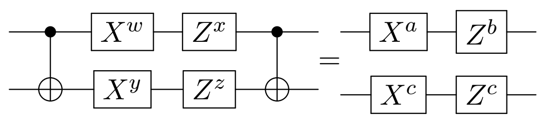
Question 7 [6 points]: Teleportation-based protocols

Suppose Alice has a qubit $|\psi\rangle$ and Bob has a qubit $|\phi\rangle$, and consider the following scenario:

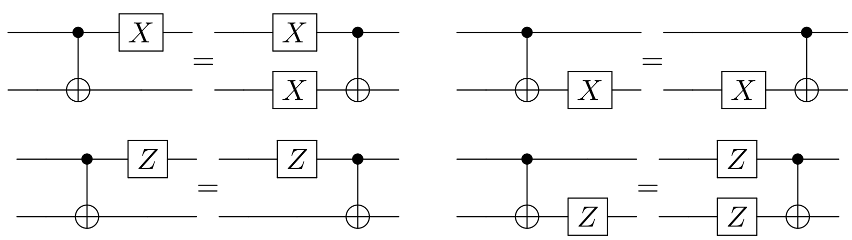
- Alice and Bob have a classical communication channel
- Alice and Bob have shared access to an unlimited source of entangled qubits
- Alice and Bob do **not** have a quantum communication channel

1. Describe a procedure by which Alice and Bob could apply a $CNOT$ gate to their pair of qubits — i.e. $CNOT(|\psi\rangle \otimes |\phi\rangle)$

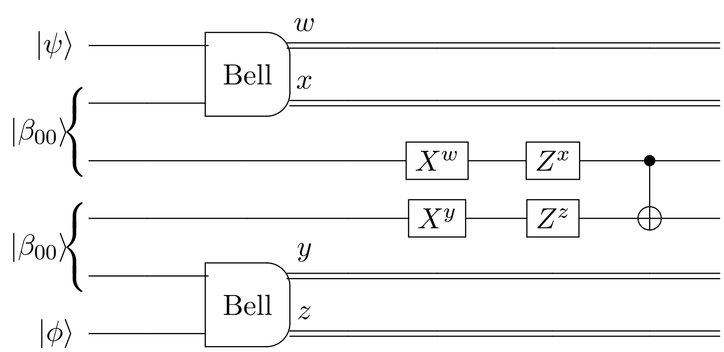
2. Find values $a, b, c, d \in \{0, 1\}$ as functions of w, x, y, z such that



   You may find the following circuit equalities useful for this question:



3. Explain why the following circuit would implement a $CNOT$ gate on the state $|\psi\rangle|\phi\rangle$



4. Let

$$|\Delta\rangle = (I \otimes CNOT \otimes I)(|\beta_{00}\rangle \otimes |\beta_{00}\rangle) = \frac{1}{2}\left(|0000\rangle + |0011\rangle + |1110\rangle + |1101\rangle\right)$$

   be a 4 qubit entangled state. Suppose Alice has the first two qubits of $|\Delta\rangle$ and Bob has the second two. Explain why the circuit below where a, b, c, d are the functions of w, x, y, z

you gave in part 3 implements a *remote* $CNOT$ between their qubits — that is, applies $CNOT(|\psi\rangle \otimes |\phi\rangle)$ *without* Alice or Bob physically teleporting their qubits to one another.